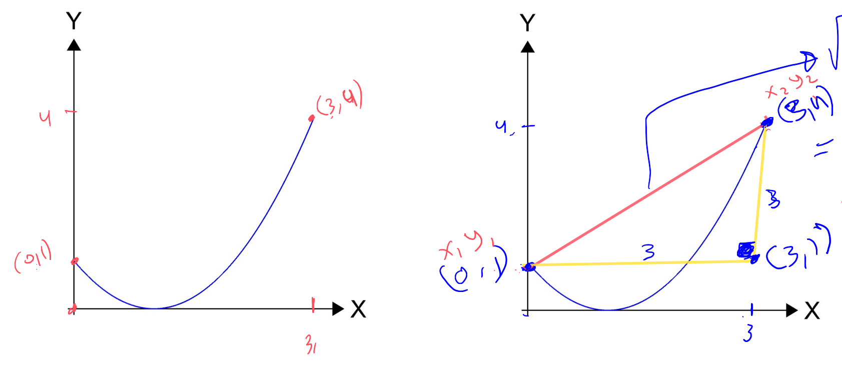
Chapter 8: Further Applications of Integration

Section 8.1: Arc Length

Objective: In this lesson, you learn

- ☐ How to define and evaluate the length of a curve defined on a finite interval as the limit of Riemann sums.
- ☐ How to define and obtain the arc length function describing the distance a particle traveled along a curve

Problem: Find the length of the arc of the parabola $y = (x-1)^2$ between the points $(0, 1)$ and $(3, 4)$?



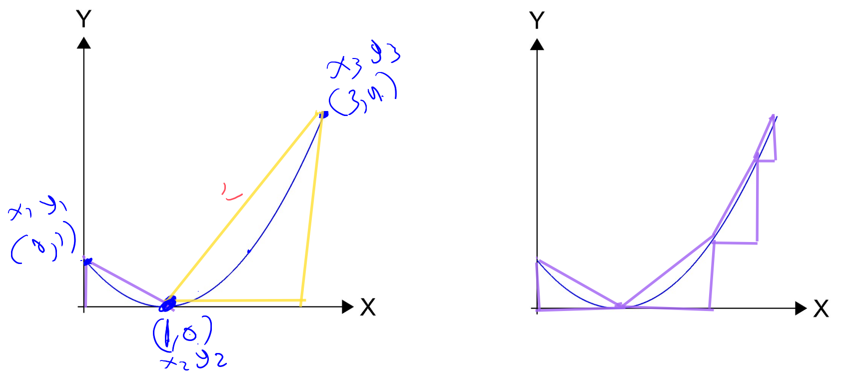
$\sqrt{3^2+3^2} = \sqrt{18}$

$= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$= \sqrt{(0-3)^2+(1-4)^2}$

$= \sqrt{3^2+3^2} = \sqrt{18}$

distance between two points

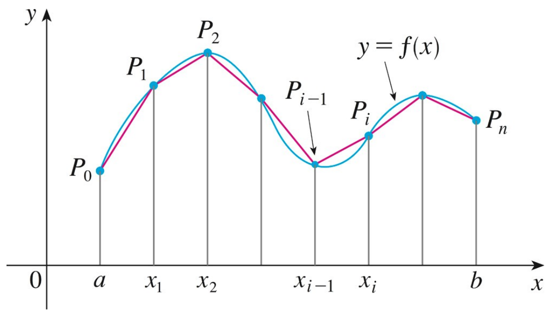


$\sqrt{(1)^2+(1)^2} + \sqrt{(2)^2+(4)^2}$

$= \sqrt{2} + \sqrt{20}$

I. Arc Length

If a curve is a polygon, then it is easy to find its length. In general, suppose that a curve C is defined by the equation $y = f(x)$, where f is continuous and $a \leq x \leq b$. Divide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and equal width Δx.
If $y_i = f(x_i)$, then the point $P_i(x_i, y_i)$ lies on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ is an approximation to C.
The approximation gets better as n increases.



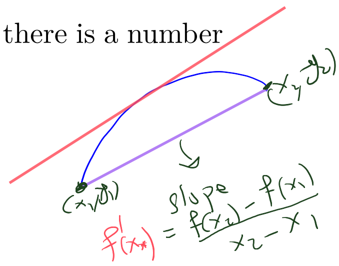
So define the length L of the curve C with the equation $y = f(x)$, $a \leq x \leq b$, as the limit of the lengths of these inscribed polygons (if the limit exists):

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

Now, if f has a continuous derivative then f is called smooth because a small change in x produces a small change in $f'(x)$. Let $\Delta y_i = y_i - y_{i-1}$, then

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

Apply the Mean Value Theorem to f on the subinterval $[x_i, x_{i-1}]$ to find that there is a number x_i^* in $[x_i, x_{i-1}]$ such that



$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}),$$

that is, $\Delta y_i = f'(x_i^*)\Delta x$. Thus,

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} = \sqrt{1 + [f'(x_i^*)]^2}\sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

Therefore,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x,$$

which, by the definition of a definite integral, is equal to

$$\int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

The integral exists because the function

$$g(x) = \sqrt{1 + [f'(x)]^2}$$

is continuous.

Restate the definition of arc length as follows:

Arc Length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

In the Leibniz notation,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If a curve has the equation $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

Example 1: Consider the curve $y = \dfrac{x^2}{2} - \dfrac{\ln x}{4}$. Calculate the length of this curve from $x = 1$ and $x = 2$.

$$f(x) = \frac{x^2}{2} - \frac{\ln x}{4}$$

$$f'(x) = x - \frac{1}{4x}$$

$$(f'(x))^2 = \left(x - \frac{1}{4x}\right)^2 = x^2 - 2 * \cancel{x} * \frac{1}{4\cancel{x}} + \frac{1}{16x^2}$$

$$= x^2 - \frac{1}{2} + \frac{1}{16x^2}$$

$$1 + (f'(x))^2 = 1 - \frac{1}{2} \quad x^2 + \frac{1}{16x^2} = \frac{1}{2} + x^2 + \frac{1}{16x^2}$$

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} = \sqrt{\left(x + \frac{1}{4x}\right)\left(x + \frac{1}{4x}\right)}$$

$$= \sqrt{\left(x + \frac{1}{4x}\right)^2} = x + \frac{1}{4x}$$

$$\int_1^2 \sqrt{1 + (f'(x))^2}\, dx = \int_1^2 x + \frac{1}{4x}\, dx = \frac{x^2}{2} + \frac{1}{4} \ln x \Big|_1^2$$

$$= \left(\frac{4}{2} + \frac{1}{4}\ln 2\right) - \left(\frac{1}{2} + \cancel{\frac{1}{4}\ln 1}\right)$$

$$= \frac{3}{2} + \frac{1}{4}\ln 2$$

Example 2: Find the length of the curve $x^2 = 4(y+4)^3$, $0 \le y \le 2$, $x > 0$?

$x = +\sqrt{4(y+4)^3}$, $x > 0$

$x = 2(y+4)^{3/2}$.

we perform integration in y from 0 to 2.

$$L = \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$\left[f(g(x))\right]' = f'(g(x)) * g'(x)$

$x = 2(y+4)^{3/2}$

$\frac{dx}{dy} = 3(y+4)^{1/2}$

$\left(\frac{dx}{dy}\right)^2 = \left(3(y+4)^{1/2}\right)^2 = 9(y+4) = 9y + 36$

$1 + \left(\frac{dx}{dy}\right)^2 = 9y + 37$

$u = 9y + 37$
$du = 9\, dy$

$$L = \int_0^2 \sqrt{9y+37}\, dy = \int_0^2 (9y+37)^{1/2}\, dy$$

$$= \int_{y=0}^{y=2} u^{1/2} \cdot \frac{du}{9} = \frac{1}{9} \frac{u^{3/2}}{3/2} \Big|_{y=0}^{y=2}$$

$$= \frac{2}{27} (9y+37)^{3/2} \Big|_0^2$$

$$= \frac{2}{27}\left((55)^{3/2} - (37)^{3/2}\right)$$

II. The Arc Length Function

It will be useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve.

Arc length function

Let a smooth curve C has the equation $y = f(x)$, $a \leq x \leq b$. Let $s(x)$ be the distance along C from the initial point $P_0(a, f(a)$ to the point $Q(x, f(x)$ then

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt,$$

called the **arc length function**

By the Fundamental Theorem of Calculus, differentiate $s(x)$ to obtain

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

which shows that the rate of change of s with respect to x is always at least 1 and is equal to 1 when $f'(x)$, the slope of the curve, is 0.

Differential of arc length

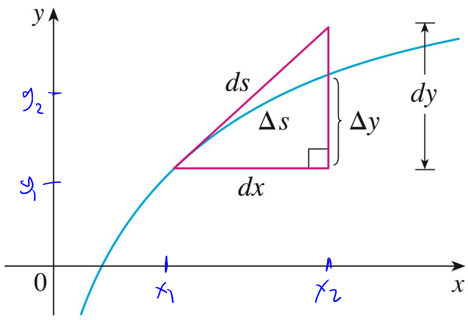
The **differential of arc length** is

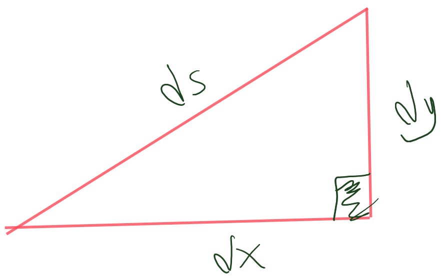
$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

$$(ds)^2 = \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx\right)^2$$

$$(ds)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right] \cdot (dx)^2$$

$$(ds)^2 = (dx)^2 + (dy)^2$$





Example 3: Find the arc length function for the curve $y = 2x^{3/2}$ with starting point $P_0(1, 2)$? What is the arc length between points $P_0(1, 2)$ and $P_1(4, f(4))$?

$$f(x) = 2x^{3/2}$$

$$f'(x) = 3x^{1/2}$$

$$(f'(x))^2 = 9x$$

$$1 + (f'(x))^2 = 9x + 1$$

$$S(x) = \int_1^x \sqrt{1 + (f'(t))^2}\, dt = \int_1^x \sqrt{9t+1}\, dt$$

$$= \int_1^x (9t+1)^{1/2}\, dt = \frac{1}{9} \left. \frac{(9t+1)^{3/2}}{3/2} \right|_1^x$$

$$S(x) = \frac{2}{27}\left[(9x+1)^{3/2} - 10^{3/2}\right]$$

the length from $P_0(1,2)$ to $P_1(4, f(4))$

$$S(4) = \frac{2}{27}\left((37)^{3/2} - 10^{3/2}\right).$$